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A spin-conformal lower bound of the first positive Dirac eigenvalue

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Abstract

Let D be the Dirac operator on a compact spin manifold M. Assume that 0 is in the spectrum of D. We prove the existence of a lower bound on the first positive eigenvalue of D depending only on the spin structure and the conformal type.

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1. Introduction

A central problem in spectral geometry is to find estimates for small eigenvalues of the classical elliptic differential operators on compact Riemannian manifolds. Among the most important are the Laplace operator, the Yamabe operator and the Dirac operator. On an n-dimensional manifold, $n \geqslant 3$, the Yamabe operator of the Riemannian manifold (M, g) is defined as

$$Y_g := 4\frac{n-1}{n-2}\Delta_g + \mathrm{scal}_g.$$

This operator plays a crucial role in the solution of the Yamabe problem, the problem of finding a metric of constant scalar curvature in a given conformal class $[g^0]$ on the manifold M (see [8] for a good

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overview). Yamabe observed that the metric $g \in [g^0]$ has constant scalar curvature if and only if g is a critical point of the functional

$$\mathcal{F}(g) := \frac{\int_M \mathrm{scal}_g \, \mathrm{d\,vol}_g}{(\mathrm{vol}(M, g))^{(n-2)/n}},$$

viewed as a functional on the metrics conformal to $[g^0]$.

This functional is bounded from below, its infimum is called the Yamabe number

$$\mu\big(M, [g^0]\big) := \inf_{g \in [g^0]} \mathcal{F}(g).$$

If $(M^n, [g^0])$ is *not* conformal to the sphere with its standard conformal structure (S^n, g_{st}), then Aubin [1] and Schoen [9] have proven that

$$\mu\big(M, [g^0]\big) < \mu\big(S^n, [g_{\mathrm{st}}]\big). \tag{1.1}$$

This estimate was the last step in solving the Yamabe problem, i.e., in showing that there actually is a metric of constant scalar curvature in each conformal class.

It is easy to see that the Yamabe number $\mu(M, [g^0])$ can also be defined in terms of the first eigenvalue $\lambda_1^Y(g)$ of the Yamabe operator Y_g:

$$\mu\big(M, [g^0]\big) = \inf_{g \in [g^0]} \big(\lambda_1^Y(g) \cdot \mathrm{vol}(M, g)^{2/n}\big).$$

In this paper we will study a similar infimum for D^2, the square of the Dirac operator. We will show that a lower bound by Lott for the first eigenvalue of D^2 generalizes to the first positive one, if D^2 has a kernel. In particular, we will obtain an estimate for the first positive eigenvalue of D^2 that only depends on the conformal structure and the spin structure.

More explicitly, let us fix a spin structure σ and a conformal class $[g^0]$ on a connected closed oriented spin manifold M of dimension $n \geqslant 2$. For a metric g conformal to g^0, let $\lambda_k(g, \sigma)$ be the kth eigenvalue of the square of the Dirac operator on (M, g, σ). Recall that for $g \in [g^0]$ we have $\lambda_k(g, \sigma) = 0$ if and only if $\lambda_k(g^0, \sigma) = 0$.

We set

$$\mathcal{Q}_k\big(M, [g^0], \sigma\big) := \inf_{g \in [g^0]} \big\{\lambda_k(g, \sigma) \cdot \mathrm{vol}(M, g)^{2/n}\big\}.$$

For $n \geqslant 3$ and under the condition that the Yamabe number $\mu(M, [g^0])$ is positive, Oussama Hijazi [4,5] established a relation of $\mathcal{Q}_1$ with the Yamabe number. He proved

$$\mathcal{Q}_1\big(M, [g^0], \sigma\big) \geqslant \frac{n}{4(n-1)} \mu\big(M, [g^0]\big). \tag{1.2}$$

For the sphere S^n with its standard conformal structure he showed equality in Eq. (1.2) and the infimum in the definition of $\mathcal{Q}_1$ is attained for constant sectional curvature metrics on S^n.

Inequality (1.2) has been extended by Christian Bär [2] to the case $n = 2$, i.e., he proved

$$\mathcal{Q}_1\big(S^2\big) = 4\pi$$

for the unique spin structure and the unique conformal class on S^2 and equality holds for the spheres of constant Gauss curvature.

John Lott [7] proved that if the Dirac operator is invertible, then

$$\mathcal{Q}_1\big(M, [g^0], \sigma\big) > 0.$$

In contrast to Hijazi's inequality, this result even holds if the Yamabe number is zero or negative.

In the present article we will extend Lott's result to the case that D is not invertible, i.e., that it has a nontrivial kernel.

Theorem 2.3. *Let D be the Dirac operator on (M, g^0, σ) and let $h := \dim\ker D$. Then*

$$\mathcal{Q}_{h+1}\big(M, [g^0], \sigma\big) := \inf_{g\in[g^0]} \big\{\lambda_{h+1}(g, \sigma) \cdot \mathrm{vol}(M, g)^{2/n}\big\} > 0.$$

It is still an open problem to find *explicit* lower bounds for $\mathcal{Q}_{h+1}$ for the case $h > 0$ and for the case $\mu(M, [g^0]) \leqslant 0$ in terms of "nice" geometric data.

In Section 3 we establish for $n \geqslant 3$ and for $n = 2$, $h = 0$ an inequality which is similar to inequality (1.1):

$$\mathcal{Q}_{h+1}\big(M, [g^0], \sigma\big) \leqslant \mathcal{Q}_1\big(S^n, [g_{\mathrm{st}}]\big).$$

For proving the inequality, we construct a conformal blow-up of a sphere in a small neighborhood of a given point. Unfortunately, it is not known to us, whether we can obtain a strict inequality in the case that $(M, [g^0])$ is not conformal to $(S^n, [g_{\mathrm{st}}])$.

2. The estimate

Let (M, g^0) be a compact connected oriented Riemannian manifold with a fixed spin structure σ.

For any smooth function $f : M \to \mathbb{R}^+$ we set

$$g_f := f^2 \cdot g^0.$$

Hence, in our notation $g_1 = g^0$. Let D_f be the Dirac operator associated to the metric g_f and the spin structure σ. The spectrum of D_f^2 is discrete and nonnegative and will be written as

$$\lambda_1(g_f, \sigma) \leqslant \lambda_2(g_f, \sigma) \leqslant \cdots,$$

where each eigenvalue appears as many times as its multiplicity.

The following transformation formula describes how the Dirac operators for conformally equivalent metrics are related. $\Sigma(M, g, \sigma)$ denotes the spinor bundle of (M, g, σ).

Proposition 2.1 [4,6]. *There is an isomorphism of vector bundles $F : \Sigma(M, g^0, \sigma) \to \Sigma(M, g_f, \sigma)$ which is a fiberwise isometry such that*

$$D_f\big(F(\Psi)\big) = F\big(f^{-(n+1)/2} D_1 f^{(n-1)/2}\Psi\big).$$

As a corollary the dimension $h := \dim\ker D_f$ of the kernel of the Dirac operator is invariant under conformal changes of the metric. However, it does depend on the choice of spin structure.

More explicitly

$$\Psi \in \ker D_1 \Leftrightarrow f^{-(n-1)/2} F(\Psi) \in \ker D_f.$$

Now let $\mathcal{C}_f$ be the orthogonal complement to $\ker D_f \subset L^2(M, g_f, \Sigma M)$. Set $\mathcal{C}_f^* := \mathcal{C}_f - \{0\}$. This orthogonal complement also transforms naturally, but clearly with another power of f

$$\Psi \in \mathcal{C}_1 \Leftrightarrow f^{-(n+1)/2} F(\Psi) \in \mathcal{C}_f.$$

In order to shorten our notation we write

$$\mathcal{Q}_k\big(M, [g^0], \sigma\big) := \inf_{g \in [g^0]} \big\{\lambda_k(g, \sigma) \cdot \mathrm{vol}(M, g)^{2/n}\big\}.$$

Theorem 2.2 [7, Proposition 1]. *If* $\dim \ker D_1 = 0$, *then* $\mathcal{Q}_1(M, [g^0], \sigma) > 0$.

We will generalize this theorem to

Theorem 2.3. *Let* (M, g^0, σ) *be any compact Riemannian spin manifold and* $h := \dim \ker D_1$. *Then*

$$\mathcal{Q}_{h+1}\big(M, [g^0], \sigma\big) > 0.$$

In other words: the theorem states that the first positive eigenvalue of the Dirac operator is uniformly bounded from below in the set of all metrics $g \in [g^0]$ with $\mathrm{vol}(M, g) = 1$.

Remark. The corresponding statement for the Laplacian on functions is false. There is a sequence of metrics (g_i) in $[g^0]$, $\mathrm{vol}(M, g_i) = 1$ such that the first positive eigenvalue $\lambda_1^{\Delta}(g_i)$ of the Laplacian on functions with respect to g_i tends to zero.

For the proof of Theorem 2.3 we follow the arguments in [7]. The proof splits into two propositions. We will write $\|\varphi\|_p$ for the L^p-norm of the spinor φ on (M, g^0, σ).

Proposition 2.4. *For the first positive eigenvalue we have the following bound*:

$$\mathcal{Q}_{h+1}\big(M, [g^0], \sigma\big) = \inf_{\varphi \in \mathcal{C}_1^*} \frac{\|\varphi\|_{2n/(n+1)}^4}{|\int \langle \varphi, |D_1|^{-1} \varphi \rangle \, \mathrm{d}\,\mathrm{vol}_1|^2}.$$

Proposition 2.5. *The right-hand side of the above formula is positive*

$$\inf_{\varphi \in \mathcal{C}_1^*} \frac{\|\varphi\|_{2n/(n+1)}^4}{|\int \langle \varphi, |D_1|^{-1} \varphi \rangle \, \mathrm{d}\,\mathrm{vol}_1|^2} > 0.$$

Proof of Proposition 2.4. We transform, substituting $\psi = f^{-(n+1)/2} F(\varphi)$

$$\begin{aligned}
\lambda_{h+1}^{-(1/2)}(g_f, \sigma) &= \sup_{\psi \in \mathcal{C}_f^*} \frac{|\int \langle \psi, D_f^{-1} \psi \rangle \, \mathrm{d}\,\mathrm{vol}_f|}{\int \langle \psi, \psi \rangle \, \mathrm{d}\,\mathrm{vol}_f} \\
&= \sup_{\varphi \in \mathcal{C}_1^*} \frac{|\int \langle f^{-(n+1)/2} F(\varphi), D_f^{-1} f^{-(n+1)/2} F(\varphi) \rangle \, \mathrm{d}\,\mathrm{vol}_f|}{\int \langle f^{-(n+1)/2} F(\varphi), f^{-(n+1)/2} F(\varphi) \rangle \, \mathrm{d}\,\mathrm{vol}_f} \\
&= \sup_{\varphi \in \mathcal{C}_1^*} \frac{|\int \langle f^{-(n+1)/2} F(\varphi), f^{-(n-1)/2} F(D_1^{-1} \varphi) \rangle \, \mathrm{d}\,\mathrm{vol}_f|}{\int \langle f^{-(n+1)/2} F(\varphi), f^{-(n+1)/2} F(\varphi) \rangle \, \mathrm{d}\,\mathrm{vol}_f}
\end{aligned}$$

$$= \sup_{\varphi \in \mathcal{C}_1^*} \frac{|\int \langle \varphi, D_1^{-1}\varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1 |}{\int f^{-1}\langle \varphi, \varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1}$$
$$= \sup_{\varphi \in \mathcal{C}_1^*} \frac{|\int \langle \varphi, |D_1|^{-1}\varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1 |}{\int f^{-1}\langle \varphi, \varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1}.$$

The Hölder inequality yields

$$\begin{aligned}\int |\varphi|^{2n/(n+1)} \, \mathrm{d}\,\mathrm{vol}_1 &= \int \big(|\varphi|^{2n/(n+1)} f^{-n/(n+1)}\big) f^{n/(n+1)} \, \mathrm{d}\,\mathrm{vol}_1 \\ &\leqslant \left(\int |\varphi|^2 f^{-1} \, \mathrm{d}\,\mathrm{vol}_1\right)^{n/(n+1)} \left(\int f^n \, \mathrm{d}\,\mathrm{vol}_1\right)^{1/(n+1)} \\ &= \left(\int |\varphi|^2 f^{-1} \, \mathrm{d}\,\mathrm{vol}_1\right)^{n/(n+1)} \mathrm{vol}(M, g_f)^{1/(n+1)}\end{aligned}$$

with equality if and only $f = c \cdot |\varphi|^{2/(n+1)}$ or $\varphi \equiv 0$. Therefore

$$\inf_{f \in C^\infty(M,\mathbb{R}^+)} \left(\mathrm{vol}(M, g_f)^{1/n} \cdot \int f^{-1}\langle \varphi, \varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1\right) \geqslant \|\varphi\|^2_{2n/(n+1)} \tag{2.1}$$

for any spinor φ.

Combining these two expressions we obtain

$$\begin{aligned}\mathcal{Q}_{h+1}\big(M, [g^0], \sigma\big) &= \inf_{f \in C^\infty(M,\mathbb{R}^+)} \inf_{\varphi \in \mathcal{C}_1^*} \frac{\mathrm{vol}(M, g_f)^{2/n} \cdot (\int f^{-1}\langle \varphi, \varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1)^2}{|\int \langle \varphi, |D_1|^{-1}\varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1 |^2} \\ &= \inf_{\varphi \in \mathcal{C}_1^*} \inf_{f \in C^\infty(M,\mathbb{R}^+)} \frac{\mathrm{vol}(M, g_f)^{2/n} \cdot (\int f^{-1}\langle \varphi, \varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1)^2}{|\int \langle \varphi, |D_1|^{-1}\varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1 |^2} \\ &\geqslant \inf_{\varphi \in \mathcal{C}_1^*} \frac{\|\varphi\|^4_{2n/(n+1)}}{|\int \langle \varphi, |D_1|^{-1}\varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1 |^2}.\end{aligned} \tag{2.2}$$

It remains to show that we have equality in (2.2). It will be sufficient to prove equality in (2.1).

If φ is nowhere vanishing, we can evaluate the integral in (2.1) for $f = |\varphi|^{2/(n+1)}$ and we see that (2.1) is actually an equality. For the case that φ vanishes somewhere this argument has to be slightly modified: Choose smooth functions $f_k : M \to \mathbb{R}^+$ such that

$$\big\| f_k - \max\big\{1/k, |\varphi|^{2/(n+1)}\big\}\big\|_{C^0} < \mathrm{e}^{-k}.$$

We conclude

$$\mathrm{vol}(M, g_{f_k})^{1/n} \cdot \int f_k^{-1}\langle \varphi, \varphi\rangle \, \mathrm{d}\,\mathrm{vol}_1 \to \left(\int |\varphi|^{2n/(n+1)}\right)^{1/n} \cdot \left(\int |\varphi|^{2n/(n+1)}\right) = \|\varphi\|^2_{2n/(n+1)},$$

and therefore we have equality in (2.1) which implies equality in (2.2). □

Proof of Proposition 2.5. We have to prove the boundedness of the map

$$\begin{aligned} H : \mathcal{C}_1 &\to L^2(\Sigma M), \\ \varphi &\mapsto |D_1|^{-1/2}\varphi, \end{aligned} \tag{2.3}$$

where $\mathcal{C}_1$ carries the $L^{2n/(n+1)}$-topology.

Let P_H be the projection operator from $L^2(\Sigma M)$ to the kernel of D_1. Clearly P_H is an infinitely smoothing operator. Hence $D_1 + P_H$ is an invertible operator on $L^2(\Sigma M)$. Trivially $(D_1 + P_H)^{-1}|_{\mathcal{C}_1^*} = (D_1|_{\mathcal{C}_1^*})^{-1}$. The symbols of D_1 and $D_1 + P_H$ are equal, therefore $D_1 + P_H$ is an elliptic pseudodifferential operator of order 1. According to Seeley [10] $|D_1 + P_H|^{-1/2} = ((D_1 + P_H)^2)^{-1/4}$ is a pseudodifferential operator of order $-1/2$.

Denote the connection Laplacian on ΣM by $\nabla^*\nabla$. Then $(\nabla^*\nabla + \mathrm{Id})^{1/4}|D_1 + P_H|^{-1/2}$ is a zero order pseudodifferential operator, and hence [11, XI, Theorem 2.2] a bounded operator $L^p(\Sigma M) \to L^p(\Sigma M)$ for any $p \in \,]1, \infty[$, in particular for $p = 2n/(n+1)$.

In other words, the map

$$\begin{aligned} A : L^{2n/(n+1)}(\Sigma M) &\to W^{1/2}_{2n/(n+1)}(\Sigma M), \\ \varphi &\mapsto |D_1 + P_H|^{-1/2}\varphi, \end{aligned} \tag{2.4}$$

is a bounded operator. Here $W^k_p(\Sigma M)$ is the Sobolev space of sections which are L^p and whose derivatives up to order k are also L^p. The Sobolev embedding theorem states that $W^{1/2}_{2n/(n+1)}(\Sigma M)$ embeds into $L^2(\Sigma M)$. Thus A is a bounded operator $L^{2n/(n+1)}(\Sigma M) \to L^2(\Sigma M)$ extending H. □

Note that we are in the boundary case of the Sobolev embedding theorem, therefore the embedding is not compact.

Remark. A variation of Proposition 2.4 is still valid if we replace the Dirac operator D_f by another elliptic differential operator T_f of order $j \in \mathbb{N}$ with the following transformation formula

$$T_f = f^{-(n+j)/2} T_1 f^{(n-j)/2}. \tag{2.5}$$

This transformation formula holds, for example, for the Yamabe operator by setting $j = 2$.

For such a T_f the formula analogous to Proposition 2.4 holds

$$\mathcal{Q}_{h+1}\big(M, [g^0], T_f\big) = \inf_{\varphi \in \mathcal{C}_1^*} \frac{\|\varphi\|^4_{2n/(n+j)}}{|\int \langle \varphi, |T_1|^{-1}\varphi\rangle \,\mathrm{d\,vol}_1|^2}.$$

The proofs runs completely analogous as for the Dirac operator.

On the other hand Proposition 2.5 is valid for $j = 1, \ldots, n-1$, hence Theorem 2.3 is also true for any operator T satisfying (2.5) with $j = 1, \ldots, n-1$.

In contrast to this, for $j = n$ the analogue of Proposition 2.5 is false. This can be easily seen by studying the Yamabe operator on surfaces. On surfaces the Yamabe operator coincides with the Laplacian on functions (see the previous remark).

3. An upper bound for $\mathcal{Q}_{h+1}$

Theorem 3.1. *Let g_{st} be the standard metric on S^n, $n = \dim M \geqslant 3$. Then*

$$\mathcal{Q}_{h+1}\big(M, [g^0], \sigma\big) \leqslant \mathcal{Q}_1\big(S^n, [g_{\mathrm{st}}]\big) = \frac{n^2}{4}\omega_n^{2/n}$$

with $\omega_n := \mathrm{vol}(S^n, g_{\mathrm{st}}) = (\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)})$.

Theorem 3.2. *Let M be a compact surface. Let g_{st} be the standard metric on S^2. Then*

$$\mathcal{Q}_1\big(M,[g^0],\sigma\big) \leqslant \mathcal{Q}_1\big(S^2,[g_{\mathrm{st}}]\big) = 4\pi.$$

Remark. If M is a compact surface of genus $\geqslant 1$, then (M, g^0, σ) may have harmonic spinors, i.e., $h \geqslant 1$. In this case it is still unknown whether $\mathcal{Q}_{h+1}(M,[g^0],\sigma) \leqslant 4\pi$.

The proofs of the two theorems run very similarly. At first, we want to give a short outline of the proof. For $n \geqslant 3$, we want to construct a metric $\hat{g}$ in the conformal class $[g^0]$ that satisfies

$$\lambda_{h+1}(\hat{g},\sigma)\cdot \mathrm{vol}(M,\hat{g})^{2/n} \leqslant \mathcal{Q}_1\big(S^n,[g_{\mathrm{st}}]\big) + \varepsilon.$$

We construct such a metric by blowing up the metric in a small neighborhood of a given point. Near this given point, the blown-up metric is C^1-close to a round sphere with a small disk removed. We construct a test spinor with support in the blown-up part and whose Rayleigh quotient is close to the first eigenvalue of the square of the Dirac operator on the sphere. If $n \geqslant 3$, then we will finally prove that this test spinor is almost orthogonal to the kernel of the Dirac operator.

We start the proof with the following proposition that constructs a suitable test spinor on the round sphere.

Proposition 3.3. *Let S^n carry the standard metric g_{st}. Fix a point p on the sphere S^n. For any $\alpha > 0$ there is a nontrivial spinor Ψ on the sphere S^n vanishing on a small neigborhood U of p and satisfying*

$$\|D\Psi\|_{L^2} \leqslant \left(\frac{n}{2}+\alpha\right)\|\Psi\|_{L^2}. \tag{3.1}$$

Proof. Recall that the spinor bundle of S^n is trivialized by $2^{[n/2]}$ Killing spinors with the Killing constant $1/2$. It is also trivialized by $2^{[n/2]}$ Killing spinors with the Killing constant $-1/2$. Killing spinors with Killing constant $\pm 1/2$ are eigenspinors with eigenvalue $\pm n/2$.

Pick Ψ_τ^+ and Ψ_τ^- a pair of Killing spinors with Killing constants $1/2$ and $-1/2$, respectively and assume that for a given $p \in S^n$ and for a given $\tau \neq 0$, $\tau \in \Sigma_p(S^n)$, we have $\Psi_\tau^+(p) = \Psi_\tau^-(p) = \tau$. Define

$$\Psi_0 := \Psi_\tau^+ - \Psi_\tau^-.$$

Let $\rho : [0,\infty[\to [0,1]$ be a smooth function with $\rho \equiv 0$ in a neighborhood of 0, $\rho \equiv 1$ in a neighborhood of $[1,\infty[$ and $0 \leqslant \rho' \leqslant 2$. For $0 \leqslant \delta < \varepsilon$ we set

$$\rho_{\delta,\varepsilon}(x) = \rho\left(\frac{d(x,p)-\delta}{\varepsilon-\delta}\right).$$

Then $|\operatorname{grad}\rho_{\delta,\varepsilon}| \leqslant \frac{2}{\varepsilon-\delta}$. See Fig. 1.

For sufficiently small $\varepsilon > 0$

$$\Psi(x) := \rho_{\delta,\varepsilon}\big(d(x,p)\big)\,\Psi_0(x)$$

defines a smooth spinor vanishing in a neighborhood of p. We calculate

$$\nabla_X\Psi = \rho_{\delta,\varepsilon}\cdot(\nabla_X\Psi_0) + X(\rho_{\delta,\varepsilon})\cdot\Psi_0,$$
$$D\Psi = \rho_{\delta,\varepsilon}\cdot(D\Psi_0) + \gamma\big(\operatorname{grad}(\rho_{\delta,\varepsilon})\big)\Psi_0.$$

Here γ denotes Clifford multiplication.

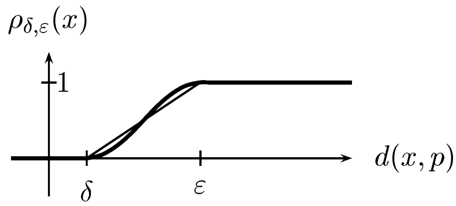

Fig. 1.

For the L^2-norms we have the triangle inequality

$$\|D\Psi\| \leqslant \|\rho_{\delta,\varepsilon} D\Psi_0\| + \big\|\gamma(\operatorname{grad}\rho_{\delta,\varepsilon})\Psi_0\big\|. \tag{3.2}$$

We use $\int_M \langle \Psi_\sigma^+, \Psi_\sigma^- \rangle = 0$ which implies that $D\Psi_0 = (n/2)\Psi_\sigma^+ + (n/2)\Psi_\sigma^-$ has the same L^2-norm as $(n/2)\Psi_0 = (n/2)\Psi_\sigma^+ - (n/2)\Psi_\sigma^-$. Hence, for small $\varepsilon > 0$ the first term of (3.2) is bounded as follows:

$$\|\rho_{\delta,\varepsilon} D\Psi_0\| \leqslant \|D\Psi_0\| = \frac{n}{2}\|\Psi_0\| \leqslant \left(\frac{n}{2} + \frac{\alpha}{2}\right)\|\Psi\|.$$

We set $C_\varepsilon := \max\{|\nabla\Psi_0(y)| \mid d(y,p) \leqslant \varepsilon\}$. Then $|\Psi_0(x)| \leqslant C_\varepsilon \cdot d(x,p)$ for $d(x,p) \leqslant \varepsilon$. Obviously C_ε is an increasing function in ε.

Using $|\operatorname{grad}\rho_{\delta,\varepsilon}| \leqslant \frac{2}{\varepsilon-\delta}$ we obtain an upper bound for the second term of (3.2)

$$\big\|\gamma(\operatorname{grad}\rho_{\delta,\varepsilon})\Psi_0\big\| \leqslant \frac{2\varepsilon}{\varepsilon-\delta} C_\varepsilon \operatorname{vol}\big(B_\varepsilon(p)\big) \leqslant \frac{\alpha}{2}\|\Psi\|$$

for sufficiently small $\varepsilon > 0$ and $\delta < \varepsilon/2$. □

Corollary 3.4. *For any $\alpha > 0$ there is $\rho > 0$ such that the following holds: If a closed Riemannian spin manifold (N,h) contains an open subset Ω such that (Ω, g) is isometric to $(S^n - \overline{B_\rho(p)}, g_{\mathrm{st}})$, then (N,h) carries a spinor Ψ supported in Ω satisfying*

$$\|D\Psi\|_{L^2} \leqslant \left(\frac{n}{2} + \alpha\right)\|\Psi\|_{L^2}.$$

Note that this corollary holds for any spin structure on N. The restriction of any spin structure to Ω is unique, as $H^1(\Omega, \mathbb{Z}_2) = \{0\}$.

Proof. Take the spinor Ψ on S^n provided by Proposition 3.3. We pull it back to Ω and obtain a compactly supported spinor. We extend it by zero to a spinor on (N,h). This spinor clearly satisfies the above properties. □

Corollary 3.5. *For any $\alpha > 0$ there are $\rho, \kappa > 0$ such that the following holds: If a closed Riemannian spin manifold (N,h) contains an open subset Ω diffeomorphic to the open disk $S^n - \overline{B_\rho(p)}$ and if the distance of the metric $h|_\Omega$ to the standard metric on the sphere g_{st} is bounded by κ in the $C^1(S^n - \overline{B_\rho(p)}, g_{\mathrm{st}})$-topology, then (N,h) carries a spinor Ψ supported in Ω satisfying*

$$\|D\Psi\|_{L^2} \leqslant \left(\frac{n}{2} + \alpha\right)\|\Psi\|_{L^2}.$$

Proof. We identify Ω with $S^n - \overline{B_\rho(p)}$. The metrics on Ω satisfy $\|h - g_{\mathrm{st}}\|_{C^1(\Omega, g_{\mathrm{st}})} \leqslant \kappa$. If κ is sufficiently small, then there is an isomorphism $S : \Sigma(\Omega, h) \to \Sigma(\Omega, g_{\mathrm{st}})$ between the spinor bundles satisfying

(a) S is fiberwise an isometry,
(b) for any spinor $\varphi \in \Gamma(\Sigma(\Omega, h))$ we have pointwise

$$\left|S(D_h\varphi) - D_{g_{\mathrm{st}}} S(\varphi)\right| \leqslant \kappa C|\varphi|,$$

where C only depends on the dimension n.

(See, for example, the claim in the proof of Proposition 7.1 in [3].)

Hence, the corollary immediately follows from the previous one. □

Proposition 3.6. *For any spin manifold (M, g^0, σ) and any $\kappa, \rho, \nu, \delta > 0$ there is a metric $\hat{g}$ conformal to g^0 satisfying*

(1) *$(M, \hat{g})$ contains an open subset Ω such that $(\Omega, \hat{g})$ is C^1-close to $(S^n - \overline{B_\rho(p)}, g_{\mathrm{st}})$. More precisely, there is a diffeomorphism $i : S^n - \overline{B_\rho(p)} \to \Omega$ such that*

$$\left\|i^*\hat{g} - g_{\mathrm{st}}\right\|_{C^1(S^n - \overline{B_\rho(p)}, g_{\mathrm{st}})} \leqslant \kappa.$$

(2) $\mathrm{vol}(M, \hat{g}) \leqslant \mathrm{vol}(\Omega, \hat{g}) + \nu$.
(3) *If $n := \dim M \geqslant 3$, then any harmonic spinor $\hat{\varphi}$ on $(M, \hat{g}, \sigma)$ satisfies*

$$\|\hat{\varphi}|_\Omega\|_{L^2(\Omega, \hat{g})} \leqslant \delta \|\hat{\varphi}\|_{L^2(M, \hat{g})}.$$

Proof. At first, in (A), we will prove the proposition under the additional assumption that the metric g^0 is Euclidean in the neighborhood of a point $q \in M$. Later on, in (B), we will show that the proposition remains true even if we drop this condition.

We fix the following notation: $B_r(q)$ is the ball around q of radius r in M with respect to the metric g^0. Similarly for $p \in S^n$ we denote by $B_r(p)$ the ball of radius r around p with respect to the standard metric g_{st} on S^n.

(A) We assume that $B_r(q)$ is flat for a small $r > 0$. We will define a suitable smooth function f and we define $\hat{g} := f^2 g^0$. This conformal change of metric will have the following properties:
  (i) On $M_1 := M - B_r(q)$ we have $\hat{g}|_{M_1} = \zeta^2 g^0|_{M_1}$, where ζ is a small positive constant.
  (ii) $(\Omega := B_{r/2}(q), \hat{g}|_\Omega)$ is isometric to $(S^n - \overline{B_{r\zeta}(p)}, g_{\mathrm{st}})$, $p \in S^n$.
  (iii) On the annular region (our connecting tube) $T := \overline{B_r(q)} - B_{r/2}(q)$, we define f^2 by a simple smooth interpolation of the two boundary factors.

We want to choose f such that its restriction to the ball $B_r(q)$ only depends on $d(x,q)^2$. See Fig. 2. On $B_{r/2}(q)$ we define

$$f_\eta(x) := \frac{\eta}{1 + \eta^2 (\frac{d(x,q)}{2})^2}.$$

$(B_{r/2}(q), f_\eta^2 g^0)$ has constant sectional curvature 1. Thus rotational symmetry implies that $(B_{r/2}(q), f_\eta^2 g^0)$ is isometric to a truncated sphere. For sufficiently small $r, \zeta > 0$, let $\eta > 0$ be the largest

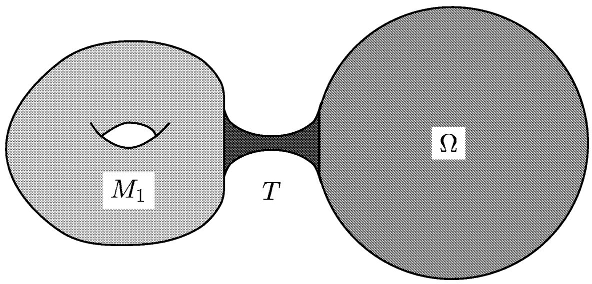

Fig. 2.

solution of the following quadratic equation in η:

$$\frac{\eta}{1+\eta^2(\frac{r}{4})^2} = 2\frac{\sin(r\zeta)}{r}.$$

Hence $(B_{r/2}(q), f_\eta^2 g^0)$ is isometric to $S^n - \overline{B_{r\zeta}(p)}$, in particular

$$f|_{\partial B_{r/2}(q)} \equiv 2\frac{\sin(r\zeta)}{r} < 2.$$

We set $f := f_\eta$ on $B_{r/2}(q)$ and $f := \zeta$ on $M_1 = M - B_r(q)$. We then interpolate f smoothly on T with $\zeta \leqslant f|_T \leqslant 2\zeta$.

Obviously

$$\eta = \max f \leqslant \frac{8}{r\sin(r\zeta)} < \frac{9}{r^2\zeta}$$

for small $r, \zeta > 0$.

If we choose r and ζ to be sufficiently small, we see that properties (1) and (2) of the proposition hold.

In order to prove (3) let

$$K := \sup\bigl\{|\varphi(x)| \mid x \in M,\ \varphi \in \ker D_1,\ \|\varphi\|_{L^2(M,g^0)} = 1\bigr\}.$$

K is finite. Let $\hat{\varphi} = f^{-(n-1)/2} F(\varphi)$ be a harmonic spinor.

$$\begin{aligned}
\|\hat{\varphi}|_\Omega\|_{L^2(\Omega,\hat{g})} &= \left| \int_\Omega f^{-n+1} \bigl|F(\varphi)\bigr|^2 f^n \,\mathrm{d}\,\mathrm{vol}_g^0 \right|^{1/2} \\
&= \bigl\|\sqrt{f}\,\varphi|_\Omega\bigr\|_{L^2(\Omega,g^0)} \\
&\leqslant \sqrt{\max f}\, K \|\varphi\|_{L^2(M,g^0)} \sqrt{\mathrm{vol}(\Omega, g^0)} \\
&\leqslant \sqrt{\frac{\max f}{\min f}}\, K \|\hat{\varphi}\|_{L^2(M,\hat{g})} \sqrt{\mathrm{vol}(\Omega, g^0)} \\
&\leqslant \frac{3}{r\zeta} K \sqrt{\frac{\omega_{n-1}}{n}} \left(\frac{r}{2}\right)^{n/2} \|\hat{\varphi}\|_{L^2(M,\hat{g})} \\
&\leqslant K' \frac{r^{n/2-1}}{\zeta} \|\hat{\varphi}\|_{L^2(M,\hat{g})}
\end{aligned}$$

for $K' := 3 \cdot 2^{-n/2} K \sqrt{\omega_{n-1}/n}$. Therefore for $n \geqslant 3$ and any $\zeta > 0$ we can choose a small $r > 0$ such that (3) of the proposition holds. Hence we have proven the theorem under the assumption that g^0 is Euclidean in the neighborhood of a point $q \in M$.

(B) Now we prove the general case. We write the metric g^0 in Gaussian normal coordinates centered in q defined on $B_r(q)$

$$g^0_{ij}(x) = \delta_{ij} + \frac{1}{3} \sum_{kl} R_{ikjl}(0) x^k x^l + \mathrm{O}\big(\|x\|^3\big). \tag{3.3}$$

Let $\chi : M \to [0, 1]$ be a C^∞-function with $\operatorname{supp} \chi \subset B_r(q)$ and $\chi|_U \equiv 1$ in a small neighborhood U of q. The arguments in (A) show that for the metric g^χ given by

$$\begin{aligned} g^\chi_{ij} &= \chi \delta_{ij} + (1 - \chi) g^0_{ij} && \text{on } B_r(p), \\ g^\chi &= g^0 && \text{otherwise,} \end{aligned}$$

there is a blow-up function $f_\chi : M \to \mathbb{R}^+$ such that (1) and (2) hold for $\hat{g}^\chi := f_\chi^2 g^\chi$. We define $\hat{g}^{(\chi)} := f_\chi^2 g^0$. It can be checked that for a suitable choice of χ properties (1) and (2) also hold for $\hat{g}^{(\chi)}$ with κ, ρ and ν replaced by 2κ, 2ρ and 2ν.

The proof of (3) in (A) carries over to the general case with some minor modifications and a slightly bigger constant K'. □

Proof of Theorem 3.1. We will prove that the inequality

$$\lambda_{h+1}(M, \hat{g}, \sigma) \operatorname{vol}(M, \hat{g})^{2/n} \leqslant \underbrace{\lambda_1(S^n, g_{\mathrm{st}})}_{n^2/4} \omega_n^{2/n} + \varepsilon \tag{3.4}$$

holds for the metric $\hat{g}$ given by Proposition 3.6, where $\varepsilon \geqslant 0$ is a small term depending on κ, ρ, ν and δ.

Corollary 3.5 states that there is a spinor Ψ on $(M, \hat{g}, \sigma)$ satisfying

$$\|D\Psi\|_{L^2} \leqslant \left(\frac{n}{2} + \alpha\right) \|\Psi\|_{L^2}.$$

Because Ψ has support in Ω we obtain the following inequality for any harmonic spinor $\hat{\varphi}$

$$(\Psi, \hat{\varphi})_{M,\hat{g}} \leqslant \|\Psi\|_{L^2(\Omega,\hat{g})} \|\hat{\varphi}\|_{L^2(\Omega,\hat{g})} \leqslant \delta \|\Psi\|_{L^2(M,\hat{g})} \|\hat{\varphi}\|_{L^2(M,\hat{g})}.$$

We apply the following trivial lemma for $A = D^2$, $\lambda = (\frac{n}{2} + \alpha)^2$ and $v = \Psi$.

Lemma 3.7. *Let A be a nonnegative self-adjoint operator on a Hilbert space $\mathcal{H}$ with pure point spectrum. Assume that a vector $v \in \mathcal{H}$ satisfies $\langle Av, v\rangle \leqslant \lambda \langle v, v\rangle$ and $\langle v, w\rangle \leqslant \delta \|v\| \|w\|$ for any $w \in \ker A$. Then there is a positive eigenvalue λ_1 of A satisfying*

$$\lambda_1 \leqslant \frac{\lambda}{1 - \delta^2}.$$

Hence, the first positive eigenvalue of D^2 on $(M, \hat{g})$ satisfies

$$\lambda_{h+1}(\hat{g}, \sigma) \leqslant \frac{(n/2 + \alpha)^2}{1 - \delta^2}.$$

Because $\operatorname{vol}(M, \hat{g})$ is bounded by $\omega_n = \operatorname{vol}(S^n, g_{\mathrm{st}})$ plus a small constant ν we obtain Eq. (3.4).

According to the result by Hijazi mentioned in the introduction (1.2)

$$\mathcal{Q}_1\big(S^n, [g_{\mathrm{st}}]\big) = \lambda_1\big(S^n, g_{\mathrm{st}}\big)\omega_n^{2/n} = \frac{n^2}{4}\,\omega_n^{2/n}. \qquad \square$$

The proof of Theorem 3.2 runs similarly. However, the argument showing that the test spinor is almost orthogonal to the space of harmonic spinors does not hold. Therefore we obtain the weaker result for $\dim M = 2$.

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